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COMPLEMENTARY PLASTIC WORK THEOREMS IN PIECEWISE-LINEAR ELASTOPLASTICITY

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Abstract—The paper deals with structures, whose constitutive laws exhibit a linear elastic range limited by independently acting yield planes, and linear or piecewise-linear hardening, or nonhardening, plastic behaviour. The consideration of the total complementary plastic work leads to a functional of the plastic strains, some properties of which are established and stated as theorems applicable to various plastic and nonlinear elastic problems.

1. INTRODUCTION

QUITE a few structural problems demand consideration of relations between generalized stresses Q_i and strains q_i, which are such that a linear elastic law is obeyed within a certain range but violated outside that range. The usual elastic work-hardening and the elastic nonhardening idealizations of material behaviour are the most relevant examples. In these cases, however, the "nonholonomic" nature of the constitutive law outside the original hookean domain is a further essential feature. Structural elements buckling in the elastic range and unilateral elastic supports are examples where the "holonomic" character of the constitutive law is preserved outside the hookean range. Following L. Finzi [1], we use here the term "holonomic" for reversible, path-independent or integrable stress–strain relations.

Let us assume that a structure, which is nonlinear for the above reasons, has to be analyzed under given loads in the domain of "small" deformations (where the influence of geometry changes on equilibrium equations is negligible). It is sometimes convenient to calculate first the linear elastic stresses due to the given external loads by assuming unlimited validity of the linear elastic law and next to evaluate the additional strains and their linear elastic consequences needed in order to comply with the actual constitutive relations. Thereafter the true structural response is determined by superposing on the linear elastic response the solution to a subsequent "corrective" nonlinear sub-problem. This approach has been proposed and emphasized by Colonnetti in classical works on elastic plastic continua [2].

For several classes of discrete cases, quadratic programming theory applied to the latter, corrective, sub-problem has led to an apparently new "finite" extremum theorem [3], which can be regarded as a counterpart concerning inelastic strains ("finite" but "small") of Ceradini's incremental or "differential" principle for plastic strain rates [4].

In this paper those results are extended to continua by a more traditional path, an alternative mechanical interpretation is proposed and some further conclusions are drawn. Possible links with other previous work on related topics are pointed out in the concluding remarks.

2. FORMULATION

Let the elastic behaviour of the elements of a continuous structure be described by the linear relation:

$$e_h = C_{hk}Q_k \qquad (h, k = 1, \ldots, m) \tag{1}$$

where Q_k are generalized stress components, and e_h elastic generalized strain components; C_{hk} are elastic constants, such that $C_{hk} = C_{kh}$ and $e_hQ_h > 0$ except for the undisturbed, stressless state. The summation convention is adopted throughout the paper for subscript indices but *not for the superscript indices*. We assume that in some or all elements, plastic strains p_h can contribute to the total strains q_h (so that $q_h = e_h + p_h$) according to the following rules.

Let Φ_i $(i = 1, \ldots, n)$ be "plastic potentials", defined, for any nonlinear element, as follows:

$$\Phi_i = N_{ih}Q_h - H^i\lambda^i - K_i, \tag{2}$$

$$\Phi_i \leq 0. \tag{3}$$

In equation (2) N_{ih} $(h = 1, \ldots, m)$, K_i, H_i are constants, the last of them (H_i) positive; the "plastic multiplier" or "activation coefficient" λ_i is related to the plastic strains through:

$$p_k = N_{ik}\lambda_i. \tag{4}$$

A point superposed on a symbol will denote derivation with respect to the time function t. This is defined as an arbitrary monotonic increasing function of the physical time, that is to say time-independent behaviour is assumed. The rates of the variables are required to satisfy the conditions:

$$\dot{\Phi}_i = 0 \text{ (hence } \dot{\lambda}_i \geq 0) \quad \text{when} \quad \Phi_i = 0 \quad \text{and} \quad N_{ih}\dot{Q}_h \geq 0 \tag{5}$$

$$\text{otherwise} \quad \dot{\lambda}_i = 0. \tag{6}$$

Relations from (2) to (6) geometrically interpreted, as usual, in the m-dimensional space with superposed co-ordinates Q_h, q_h, mean that there exist there n yield planes whose equations are represented by relations (2) for Φ_i $(Q_1, \ldots, Q_m) = 0$. These yield planes: (i) define the linear–elastic domain as the polyhedron where $\Phi_i < 0$ simultaneously for all $i = 1, \ldots, n$; (ii) *translate* together with the stress point Q $(Q_1, \ldots, Q_m)$ when the stress point reaches and "activates" some of them moving outward with respect to the elastic domain; (iii) by yielding independently of each other, they supply plastic strain contributions represented by vectors which are directed as their outward normal vectors $\vec{N}_i \equiv [N_{i1}, \ldots, N_{im}]$.

To sum up, the element behaviour assumed is elastic–plastic, with piecewise-linear yield surfaces, linear or piecewise linear hardening, obeying the normality property and Koiter's hypothesis of noninteracting yielding modes at singular points [5].

The nonholonomic character of the plastic behaviour is completely expressed by the flow rules (5), (6) only. If these rules are replaced by the following requirements:

$$\Phi_i = 0 \text{ (hence } \lambda_i \geq 0) \quad \text{when} \quad N_{ij}Q_j - K_i \geq 0, \tag{7}$$

$$\text{otherwise} \quad \lambda_i = 0 \text{ (hence } \Phi_i < 0) \tag{8}$$

one obtains the corresponding "holonomic" constitutive laws.

The holonomic laws formulated by relations from (1) to (4) supplemented by the rules (7) and (8), either reflect an additional simplifying hypothesis on the element behaviour, as in the "deformation theories" of plasticity, or result from the integration with respect to the time function t of the flow-rules (5), (6), when plastic yielding "progresses regularly" along the loading history (i.e. when no unloading occurs from a yield plane after it has been activated). In the former case, a solution obtained on the basis of (7), (8) instead of (5), (6), will be called here "holonomic solution", in contrast to the actual nonholonomic one, obtainable on the basis of relations (5), (6) by following step-by-step the whole loading history of the structure. Sometimes, as in the examples mentioned in the introduction, the holonomic constitutive laws of the preceding type can be used as a piecewise linearized description of a nonlinear but path-independent mechanical behaviour [3]. These cases are implicitly covered by the analysis which follows although the term "plastic", referred both to potentials Φ_i and strains p_h, becomes, of course, not appropriate for them; explicit reference to these cases in more suitable terms will be made in the conclusions only.

The response of the continuum considered to given loads on the hypothesis of unlimited linear–elastic behaviour according to equation (1), is supposed to have been preliminarily calculated by means of one of the classical elasticity methods. The superscript E will denote any quantity obtained under this hypothesis (e.g. Q_h^E, e_h^E). We shall be concerned here after only with the determination of the set of n scalar functions λ_i which define the distribution of the inelastic strains p_h.

3. ANALYSIS

3.1. Let a continuum, whose constituents obey the nonholonomic stress–strain laws considered in Section 2, be subjected, at instant τ, to surface tractions T_l on the unconstrained region S_T of its boundary and to body forces F_l throughout its volume V (index l refers to the axes of a reference system for the whole structure).

The actual displacements u_l at any instant t may be considered as the sum of the displacements u_l^E in a purely linear–elastic regime under the same load condition, and of the path-dependent plastic remainder $u_l^P = u_l - u_l^E$. Thus the quantity:

$$L_{CP} = \int_{S_T} \mathrm{d}S \int_0^\tau \dot{T}_l(t) u_l^P(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{F}_l(t) u_l^P(t)\,\mathrm{d}t \tag{9}$$

may be referred to as "*total complementary plastic work*" performed during the loading history from the undisturbed state at $t = 0$ up to $t = \tau$.

Let Q_h^S denote the self-stresses generated by the plastic strains p_h; let the symbol e_h^S denote the elastic strains related to Q_h^S through equation (1).

The actual stresses Q_h and the actual total strains q_h at any t may be conceived as split into the previously defined addends:

$$Q_h = Q_h^E + Q_h^S, \qquad q_h = e_h + p_h = e_h^E + e_h^S + p_h.$$

The displacement field u_l and the strain set q_h are compatible and the same holds for u_l^E and e_h^E; also the strains $p_h + e_h^S$ correspond to a compatible deformation, namely to the one defined by the field of displacements u_l^P at the same time. Apply the principle of virtual work by associating to the compatible extensive quantities $u_l^P, p_h + e_h^S$, the equilibrated system of external force increments $\dot{T}_l\,\mathrm{d}t$, $\dot{F}_l\,\mathrm{d}t$, and stress increments $\dot{Q}_h\,\mathrm{d}t$. Thus, if the displacement field is supposed to be continuous, the right side of equation (9) can be rewritten in the form:

$$L_{CP} = \int_V \mathrm{d}V \int_0^\tau \dot{Q}_h(t)p_h(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{Q}_h(t)e_h^S(t)\,\mathrm{d}t. \tag{10}$$

On the right side of (10), let us introduce equation (4) into the first term. For the second term, taking into account that $\dot{Q}_h = \dot{Q}_h^E + \dot{Q}_h^S$ and that $\dot{Q}_h^E e_h^S = \dot{e}_k^E Q_k^S$ due to the symmetry of the matrix $\{C_{hk}\}$ in equation (1), let us apply the principle of virtual work to the compatible deformation $\dot{e}_h^E, \dot{u}_l^E$ and to the self-equilibrated stresses $\dot{Q}_k^S$. Thus, equation (10) becomes:

$$L_{CP} = \int_V \mathrm{d}V \int_0^\tau \dot{Q}_h(t)N_{ih}\lambda_i(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{Q}_h^S(t)e_h^S(t)\,\mathrm{d}t \tag{11}$$

whence, substituting into (11) equation (2) differentiated with respect to time:

$$L_{CP} = \int_V \mathrm{d}V \int_0^\tau \lambda_i(t)\dot{\Phi}_i(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau [H_i\lambda_i(t)\dot{\lambda}_i(t)\,\mathrm{d}t + \dot{Q}_h^S(t)e_h^S(t)]\,\mathrm{d}t. \tag{12}$$

Integrate now over the time interval τ: the first integral by parts and taking into account the flow rule (5), which implies that $\dot{\lambda}_i\Phi_i = 0$ at any instant t. We obtain:

$$L_{CP} = \int_V \Phi_i\lambda_i\,\mathrm{d}V + \tfrac{1}{2}\int_V (H_i\lambda_i^2 + Q_h^S e_h^S)\,\mathrm{d}V. \tag{13}$$

The second integral in (13) is always positive and depends only on the final distribution of plastic strains. The first integrand is non-positive, since $\Phi_i \leq 0$, equation (3), and $\lambda_i \geq 0$ as a consequence of (5) and (6). Therefore:

$$L_{CP} \leq \tfrac{1}{2}\int_V H_i\lambda_i^2\,\mathrm{d}V + \tfrac{1}{2}\int_V Q_h^S e_h^S\,\mathrm{d}V. \tag{14}$$

The first integral vanishes in (13), and the equality sign holds in (14), only if, at $t = \tau$, the stress point of any element lies on all yield planes which have been activated; this condition is satisfied when the yielding process is regularly progressive. Note in passing that the second integral of (14), which recurrently appears in the subsequent formulae, represents the *elastic strain energy* connected with the plastic strains, i.e. "*stored*" in the structure as their consequence.

Substituting equation (2) into (13), L_{CP} is expressed as a functional of the given external forces (through their linear consequence Q_h^E) and of the final plastic strain field (through the scalar functions λ_i which define it and through its linear consequences Q_h^S, e_h^S):

$$L_{CP} = \int_V (Q_h^E N_{ih} - K_i)\lambda_i\,\mathrm{d}V - \tfrac{1}{2}\int_V (H_i\lambda_i^2 + Q_h^S e_h^S)\,\mathrm{d}V. \tag{15}$$

3.2. We shall now calculate the total complementary plastic work for the same structure and loading condition, but attributing a holonomic character to the constitutive laws, namely replacing the incremental rules (5), (6) with the finite rules (7), (8). A superscript 0 will label the variables corresponding to the solution in this case, i.e. to the "holonomic solution".

The same path of reasoning as above leads to an expression completely analogous to equation (12). The present holonomic hypothesis, however, requires the first integrand of (12) to vanish identically, since $\Phi_i^0(t) = 0$ and, hence $\dot{\Phi}_i^0(t) = 0$, as long as $\lambda_i^0 \neq 0$, by virtue of relations (7), (8). Therefore, after the integration over $\overline{0\tau}$, we obtain:

$$L_{CP}^0 = \tfrac{1}{2}\int_V H_i \lambda_i^{0^2}\, dV + \tfrac{1}{2}\int_V Q_h^{S^0} e_h^{S^0}\, dV \tag{16}$$

whence, by using equation (2) and taking into account that now $\Phi^{i^0}\lambda^{i^0} = 0$, this alternative form is attained:

$$L_{CP}^0 = \tfrac{1}{2}\int_V (Q_i^E N_{ih} - K_i)\lambda_i^0\, dV. \tag{17}$$

A combination of (16) and (17) gives, in full analogy to (15):

$$L_{CP}^0 = \int_V (Q_h^E N_{ih} - K_i)\lambda_i^0\, dV - \tfrac{1}{2}\int_V (H_i \lambda_i^{0^2} + Q_h^{S^0} e_h^{S^0})\, dV. \tag{18}$$

3.3. Consider now an expression analogous to (15) and (18), but written on the basis of a plastic strain field p_h^* defined through the normality relation (4) by *arbitrarily chosen* non-negative activation functions $\lambda_i^* \geq 0$ and on the basis of its linear consequences Q_h^{S*} and e_h^{S*}:

$$\Psi(\lambda^*) = \int_V (Q_h^E N_{ih} - K_i)\lambda_i^*\, dV - \tfrac{1}{2}\int_V H_i \lambda_i^{*^2}\, dV - \tfrac{1}{2}\int_V Q_h^{S*} e_h^{S*}\, dV. \tag{19}$$

By means of the principle of virtual work, we have:

$$\int_V Q_h^{S*} e_h^{S*}\, dV = -\int_V Q_h^{S*} p_h^*\, dV. \tag{20}$$

If x, ξ are symbols for space co-ordinates in V and if $z_{hk}(x, \xi)$ are the relevant influence functions, one may express explicitly as follows the dependence of the self-stresses Q_h^{S*} on the plastic strains p_k^* which give rise to them as dislocations (in the sense of Volterra and Love) operating on the structure in linear elastic conditions:

$$Q_h^{S*}(x) = \int_V z_{hk}(x, \xi) p_k^*(\xi)\, dV. \tag{21}$$

Through (20) and (4) expression (19) becomes:

$$\begin{aligned}\Psi(\lambda^*) = {} & \int_V (Q_h^E N_{ih} - K_i)\lambda_i^*\, dV - \tfrac{1}{2}\int_V H_i \lambda_i^{*^2}\, dV \\ & + \tfrac{1}{2}\iint_V \lambda_i^*(x) N_{ih}(x) z_{hk}(x, \xi) N_{jk}(\xi)\lambda_j^*(\xi)\, dV\, dV.\end{aligned} \tag{22}$$

In this form Ψ is explicitly expressed as a *quadratic functional of the non-negative functions* $\lambda_i^*(x)$ $(i = 1, \ldots, n)$ defined over V. As has been seen, for the actual solution λ_i and for the holonomic solution λ_i^0, the functional Ψ represents the total complementary plastic work L_{CP}, equation (15), and L_{CP}^0, equation (18), respectively. For an arbitrary set $\lambda_i^* \geq 0$, $\Psi(\lambda^*)$ may be given the mechanical interpretation that follows.

Let T_l^*, F_l^* be forces capable of generating the arbitrarily chosen plastic strains p_i^* through any given loading path according to the nonholonomic stress–strain relations.

The total complementary plastic work performed along this path can be written in form (13):

$$L_{CP}^{*} = \int_V \Phi_i^{*}\lambda_i^{*}\,\mathrm{d}V + \tfrac{1}{2}\int_V (H_i\lambda_i^{*2} + Q_h^{S*}e_h^{S*})\,\mathrm{d}V. \tag{23}$$

Let us consider the virtual work done by the above starred external forces for the corresponding plastic displacements u_l^{P*}. This "*virtual plastic work*", denoted by L_{VP}^{*}, can be expressed as follows, subsequently making use of the principle of virtual work and of equations (1) and (2):

$$\begin{aligned} L_{VP}^{*} &= \int_{S_T} T_l^{*}u_l^{P*}\,\mathrm{d}S + \int_V F_l^{*}u_l^{P*}\,\mathrm{d}V = \int_V Q_h^{*}(p_h^{*} + e_h^{S*})\,\mathrm{d}V \\ &= \int_V Q_h^{*}p_h^{*}\,\mathrm{d}V + \int_V Q_h^{S*}e_h^{S*}\,\mathrm{d}V = \int_V (\Phi_i^{*}\lambda_i^{*} + H_i\lambda_i^{2} + K_i\lambda_i^{*})\,\mathrm{d}V + \int_V Q_h^{S*}e_h^{S*}\,\mathrm{d}V. \end{aligned} \tag{24}$$

Let us now consider the *virtual plastic work* L_{VP} done by the *actual* given loads for the same compatible deformation as above, described by the displacement field u_l^{P*}. By means of the principle of virtual work† and of equation (4), we obtain:

$$L_{VP} = \int_S T_l u_l^{P*}\,\mathrm{d}S + \int_V F_l u_l^{P*}\,\mathrm{d}V = \int_V Q_h^{E}p_h^{*}\,\mathrm{d}V = \int_V N_{ih}Q_h^{E}\lambda_i^{*}\,\mathrm{d}V. \tag{25}$$

The difference $L_{VP}^{D} = L_{VP} - L_{VP}^{*}$ represents the virtual work performed, again for the deformation due to the trial plastic strains p_h^{*}, by the external force set $T_l - T_l^{*}$, $F_l - F_l^{*}$, i.e. by the difference between the actually given load system and the force system which would produce the plastic strains p_h^{*}.

The algebraic sum $L_{CP}^{*} + L_{VP} - L_{VP}^{*}$, by introducing equation (23) and the last forms of equations (24) and (25), turns out to identify with the right side of equation (19). Therefore:

$$\Psi(\lambda^{*}) = L_{CP}^{*} + L_{VP}^{D}. \tag{26}$$

On the ground of the preceding definitions for L_{CP}^{*} and L_{VP}^{D}, equation (26) specifies the mechanical meaning of the functional $\Psi(\lambda^{*})$ for any set $\lambda_i^{*} \geq 0$. When the λ_i^{*}-set identifies either with the nonholonomic solution λ_i or with the holonomic solution λ_i^{0}, equation (26) is reduced either to $\Psi(\lambda) = L_{CP}$ or to $\Psi(\lambda^{0}) = L_{CP}^{0}$ respectively.

3.4. We shall now prove two inequalities relating to the functional $\Psi(\lambda^{*})$. Given the linear plastic response Q_i^{E}, let $\Delta(\quad)$ indicate the difference between the value the argument assumes for the holonomic solution λ_i^{0} relative to the assigned loads, and the value it assumes for a set of non-negative functions $\lambda_i^{*} \geq 0$ $(i = 1, \ldots, n)$ arbitrarily chosen over V. Where necessary superscripts 0 and * will designate quantities pertaining to one or the other case. From equations (15) and (18) we obtain:

$$\begin{aligned} \Delta\Psi = \Psi(\lambda^{0}) - \Psi(\lambda^{*}) &= \int_V (N_{ih}Q_h^{E} - K_i)\Delta\lambda_i\,\mathrm{d}V \\ &- \int_V \tfrac{1}{2}H_i\Delta(\lambda_i^{2})\,\mathrm{d}V + \int_V \tfrac{1}{2}\Delta(Q_h^{S}p_h)\,\mathrm{d}V. \end{aligned} \tag{27}$$

† The symmetry of the elastic constant matrix and the principle of virtual work imply that:

$$\int_V Q_h^{E}e_h^{S*}\,\mathrm{d}V = \int_V Q_h^{S*}e_h^{E}\,\mathrm{d}V = 0.$$

Through the equation:

$$\int_V Q_h^{S*} p_h^0 \,\mathrm{d}V = \int_V Q_h^{So} p_h^* \,\mathrm{d}V† \tag{28}$$

and further, through equation (4), we may write:

$$\begin{aligned}\int_V \Delta(Q_h^S p_h)\,\mathrm{d}V &= -\int_V \Delta Q_h^S \Delta p_h \,\mathrm{d}V + 2\int_V Q_h^{S^o} \Delta p_h \,\mathrm{d}V \\ &= -\int_V \Delta Q_h^S \Delta p_h \,\mathrm{d}V + 2\int_V Q_h^{S^o} N_{hi} \Delta\lambda_i \,\mathrm{d}V.\end{aligned} \tag{29}$$

Substituting into equation (27), equation (29) and the identity

$$\Delta(\lambda_i^2) = -(\Delta\lambda_i)^2 + 2\lambda^{i^o}\Delta\lambda^i,$$

equation (27) acquires the form:

$$\Delta\Psi = \int_V [(Q_h^{S^o} + Q_h^E)N_{hi} - H^i\lambda_i^0 - K_i]\Delta\lambda_i \,\mathrm{d}V + \tfrac{1}{2}\int_V [H_i(\Delta\lambda_i)^2 - \Delta Q_h^S \Delta p_h]\,\mathrm{d}V. \tag{30}$$

Finally, by means of equation (2), taking into account that $Q_h^{S^o} + Q_h^E = Q_h^0$, and by means of a virtual work equation analogous to (20), from equation (30) we obtain:

$$\Delta\Psi = \int_V -\Phi_i^0\lambda_i^* \,\mathrm{d}V + \int_V \Phi_i^0\lambda_i^0 \,\mathrm{d}V + \tfrac{1}{2}\int_V H_i(\Delta\lambda_i)^2 \,\mathrm{d}V + \tfrac{1}{2}\int \Delta Q_h^S \Delta e_h^S \,\mathrm{d}V. \tag{31}$$

In the right side of (31) the first integrand is non-negative, as Φ_i^0 and λ_i^* cannot have the same sign; the second is zero, since in the holonomic solution either Φ_i^0 or λ_i^0 must be zero for any element and any yield plane i; unless $\lambda_i^0 = \lambda_i^*$, the third integral is positive as long as $H_i > 0$ for any i (strain-hardening behaviour); the fourth integrand cannot be negative because it represents "stored" elastic energy, as has been previously noticed (Section 3.1). These remarks lead to the conclusion that:

$$\Psi(\lambda^0) \geq \Psi(\lambda^*) \quad \text{for any} \quad \lambda_i^*(x) \geq 0 \qquad (i = 1, \ldots, n). \tag{32}$$

The plastic strain field actually produced by the given loads in the loading history is defined by functions λ_i which certainly belong to the class of all nonnegative functions $\lambda_i^* \geq 0$ over V. Therefore the preceding inequality implies also that:

$$\Psi(\lambda) \leq \Psi(\lambda^0). \tag{33}$$

In the inequalities (32) and (33) the equality sign holds if and only if $\lambda_i^* \equiv \lambda_i^0$, or, respectively, $\lambda_i^0 \equiv \lambda_i$.

† In the virtual work equations analogous to (20)

$$\int_V Q_h^{S*} p_h^0 \,\mathrm{d}V + \int_V Q_h^{S*} e_h^{S^o} \,\mathrm{d}V = 0; \qquad \int_V Q_h^{S^o} p_h^* \,\mathrm{d}V + \int_V Q_h^{S^o} e_h^{S*} \mathrm{d}V = 0$$

the second integrals are equal because of the symmetry of the elastic constants $C_{hk} = C_{kh}$; equation (28) follows.

4. CONCLUSIONS

4.1. Considering once again the problem of finding the plastic strains formulated in Section 2, let us deal first with nonholonomic constitutive laws of the type described, viz. elastic plastic continua in the framework of an incremental theory. In this context inequality (32) may be expressed in the following terms:

(Theorem 1) "*For given loads, of the admissible plastic strain fields the one corresponding to the holonomic solution maximizes the functional* Ψ, *equation* (22), *which represents in both the actual and the holonomic solutions the total complementary plastic work pertaining to them.*"

The term "admissible" is used to draw attention to the fact that any trial plastic strain set must be defined by nonnegative scalars λ_i^*, i.e. must comply with the outward normality rule expressed by equation (4) and by the sign restriction $\lambda_i^* \geq 0$.

This theorem, like the other "finite" extremum theorems so far formulated in plasticity, characterises the solution of the problem merely within the framework of deformation theory. The holonomic solution is the actual one only if the loading history is such that any activated yield surface contains the stress point, or, in particular, such that plastic yielding progresses regularly. When the loading path is particularly simple, e.g. when all loads are proportional to a monotonically increasing parameter, sometimes, but not always, we can expect these conditions to be satisfied.

In the general case inequality (33) allows a comparison between the actual and the holonomic solution, since it may be expressed, on the ground of the above analysis, by the following corollary:

(Theorem 2) "*For given loads, the total complementary plastic work pertaining to the holonomic solution is an upper bound on the total complementary plastic work of the actual path-dependent solution whatever the loading path may be.*"

Since the total complementary elastic work is equal in both cases, this statement holds even if the term "plastic" is omitted and the *total complementary energy* as a whole is referred to.

It is worth noting that inequality (14), Section 3.1, merely supplies a rather obvious upper bound on the total complementary plastic work that can be performed for *given final plastic strains*. It can be expressed alternatively as follows: "*If a distribution of plastic strains can be reached through a loading path of regular progression, this path maximizes, within the class of all loading paths leading to that distribution, the total complementary plastic work.*"

4.2. Consider now truly holonomic constitutive laws with the restrictive characteristics specified in Section 2, i.e. referring to continua whose elements behave elastically, according to nonlinear but piecewise linearized laws.

For this case the extremum property of $\Psi(\lambda^*)$, inequality (32), applies to the *true* solution and can be restated in more appropriate terms as follows:

(Theorem 3) "*For given loads, of all fields of 'admissible corrective' strains, the actual solution maximizes the functional* Ψ, *equation* (22), *which represents at the solution the total complementary work relative to the corrective strains.*"

Here again "admissible" must be understood in the sense specified for statement 1. The term "corrective" replaces plastic for the strains p_h which represent the unknowns of the problem.

We note in passing that the class of trial strain distributions still includes all the distributions corresponding to all functions sets $\lambda_i^* \geq 0$ $(i = 1, \ldots, n)$ defined over V and does not need to meet the restrictions imposed by the holonomy of the stress–strain laws.

4.3. The remarks which follow extend the applicability of the preceding conclusions and indicate their links with previous work.

(a) Only surface and body forces have been considered in discussing the mechanical interpretation of functional Ψ, but only the linear elastic response to them is present in the expression of $\Psi(\lambda^*)$ as the known functions Q_h^E. If the structure is acted upon differently (displacements imposed by moving constraints, temperature changes etc.) the resulting stresses Q_h^E may be always attributed to equivalent loads and the theorems formulated apply unaltered.

(b) The discussion has been carried out for simplicity with reference to hardening behaviour outside the linear–elastic range. If, however, the hardening coefficients H_i introduced in equation (2), are taken all zero, then merely the equality sign must hold in the last relations at (5) and (7) and the functional $\Psi(\lambda^*)$ looses the term containing H_i. On the right side of equation (31) the third integral vanishes but the fourth is still non-negative, and, consequently, the properties established for $\Psi(\lambda^*)$ still hold.

Hence the above theorems cover the elastic perfectly plastic behaviour as long as it is understood that in this case neither the holonomic nor the nonholonomic solution is necessarily unique or even exists. Equation (16) shows that, in this case, the total complementary plastic work for the holonomic solution equals the elastic strain energy connected with the plastic strains which correspond to the holonomic solution.

Within the particular context of perfect plasticity, theorem 2 can also be directly deduced from Hodge's theorem of bounding complementary energy established by a quite different approach [6].

(c) A negative coefficient H_i, along with the inversion of the last inequality in (5) and (7) for the same index i, would represent a softening or unstable behaviour at the corresponding boundary face of the linear elastic range, as an alternative to a stable stress path of "elastic unloading". Section 3.4 shows that allowance for such occurrence by no means affects the validity of the theorems established, as long as the sum of the last two integrals in equation (31) remains non-negative definite, i.e. provided that the quadratic part of functional $\Psi(\lambda^*)$, as defined by equation (22), remains non-positive definite. Thus the present conclusions appear to be subjected not to Drucker's stability postulate [7], but to a weaker overall condition, similar to those discussed in [8] for the stability of sets of situations, with reference to truss-like structures.

(d) The holonomic constitutive laws considered in the paper can be adopted as a piecewise linearization of nonlinear elastic stress–strain relations, particularly for structures consisting of one-component elements ($m = 1$), e.g. for beams analyzed in terms of bending moment and curvature as generalized stress and strain variables.

Because of Hoff's analogy between nonlinear elastic and steady creep problems [9], theorem 3 may be applied also to the latter ones.

(e) Theorems 1 and 3 reduce the holonomic solution of the problems formulated in Section 2, to the sign constrained optimization of the quadratic functional $\Psi(\lambda^*)$ expressed by equation (22). This optimization might be efficiently carried out by means of the analytical methods proposed by Villaggio [10] for constrained solutions. For the parallel discrete problems some quadratic programming procedures have been applied by the writer [11], and proved to be fit for various practical cases.

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Абстракт—Работа занимается конструкциями, которых законы состояния проявляют линейный упругий предел, при независимо действующих плоскостях течения и линейное пластическое поведение с упрочнением или без упрочнения. Рассмотрение полной, дополнительной пластической работы приводит к функционалу пластических деформаций. Установливаются некоторые свойства этих деформаций и формулируются их в качестве теорем, применимых к разным пластическим и нелинейным упругим задачам.